Exercise 64

Find the limits as $x \to \infty$ and as $x \to -\infty$. Use this information, together with intercepts, to give a rough sketch of the graph as in Example 12.

$$y = x^2(x^2-1)^2(x+2)$$

Solution

To find the y-intercept, plug in $x = 0$ to the function.

$$y = 0^2(0^2-1)^2(0+2) = 0$$

Therefore, the y-intercept is $(0,0)$. To find the x-intercept(s), set $y = 0$ and solve the equation for x.

$$x^2(x^2-1)^2(x+2) = 0$$

$$x^2(x+1)^2(x-1)^2(x+2) = 0$$

$$x = 0 \quad \text{or} \quad x = -1 \quad \text{or} \quad x = 1 \quad \text{or} \quad x = -2$$

Therefore, the x-intercepts are $(0,0)$ and $(-1,0)$ and $(1,0)$ and $(-2,0)$. Calculate the limit of the function as $x \to \pm\infty$.

$$\begin{aligned}
\lim_{x\to\infty} y &= \lim_{x\to\infty} x^2(x^2-1)^2(x+2) \\
&= \lim_{x\to\infty} x^2 \cdot x^4 \left(1 - \frac{1}{x^2}\right)^2 \cdot x \left(1 + \frac{2}{x}\right) \\
&= \lim_{x\to\infty} x^7 \left(1 - \frac{1}{x^2}\right)^2 \left(1 + \frac{2}{x}\right) \\
&= \lim_{x\to\infty} \frac{\left(1 - \frac{1}{x^2}\right)^2 \left(1 + \frac{2}{x}\right)}{\frac{1}{x^7}} \\
&= \frac{(1-0)^2(1+0)}{0} \\
&= \infty
\end{aligned}$$

In the second limit, make the substitution, $u = -x$, so that as $x \to -\infty$, $u \to \infty$.

$$\begin{aligned}
\lim_{x\to-\infty} y &= \lim_{u\to\infty} (-u)^2[(-u)^2 - 1]^2(-u+2) \\
&= \lim_{u\to\infty} u^2(u^2-1)^2(-u+2) \\
&= \lim_{u\to\infty} u^2 \cdot u^4 \left(1 - \frac{1}{u^2}\right)^2 \cdot u \left(-1 + \frac{2}{u}\right) \\
&= \lim_{u\to\infty} u^7 \left(1 - \frac{1}{u^2}\right)^2 \left(-1 + \frac{2}{u}\right) \\
&= \lim_{u\to\infty} \frac{\left(1 - \frac{1}{u^2}\right)^2 \left(-1 + \frac{2}{u}\right)}{\frac{1}{u^7}} \\
&= \frac{(1-0)^2(-1+0)}{0} \\
&= -\infty
\end{aligned}$$

Below is a graph of the function versus x.